MATH 5061 Problem Set 2[1]

Due date: Feb 10, 2021

Problems: (Please hand in your assignments via Blackboard. **Late submissions will not be accepted.**)

Throughout this assignment, we use M to denote a smooth n-dimensional manifold unless otherwise stated.

1. Describe how the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ defined by
$$F(x, y, z) = (x^2 - y^2, xy, xz, yz)$$
gives rise to an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

2. Prove *Jacobi identity*: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any $X, Y, Z \in \Gamma(TM)$.

3. Let $\phi : M \to N$ be a diffeomorphism. Show that
$$\phi_*([X, Y]) = [\phi_* X, \phi_* Y] \qquad \text{for any } X, Y \in \Gamma(TM).$$
Can you use this to give another proof of the Jacobi identity in (2)?

4. Let α be a $(0, q)$-tensor on M, $X, Y_1, \cdots, Y_q \in \Gamma(TM)$ be vector fields. Show that
$$(\mathcal{L}_X \alpha)(Y_1, \cdots, Y_q) = X(\alpha(Y_1, \cdots, Y_q)) - \sum_{i=1}^{q} \alpha(Y_1, \cdots, Y_{i-1}, [X, Y_i], Y_{i+1}, \cdots, Y_q).$$

5. Let $p : \mathbb{S}^{2n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the canonical projection map where $\iota : \mathbb{S}^{2n+1} \to \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ is the unit sphere.
   (a) Show that there exists a 2-form $\omega \in \Omega^2 \mathbb{CP}^n$ such that
   $$p^* \omega = \iota^* \left(\sum_{k=0}^{n} dz^k \wedge d\bar{z}^k \right)$$
   where $dz^k = dx^k + i dy^k$ and $d\bar{z}^k = dx^k - i dy^k$.
   (b) Prove that $\omega \in \Omega^2 \mathbb{CP}^n$ is invariant under the natural action of $U(n+1)$ on $\mathbb{CP}^n$.
   (c) Prove that $\omega^k \in \Omega^{2k} \mathbb{CP}^n$ is non-zero and also invariant under the natural action of $U(n+1)$ on $\mathbb{CP}^n$.

[1] Last revised on February 4, 2021